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THE CALCULATION OF A FIRST APPROXIMATION FOR SYSTEMS WITH STRONGLY NON-LINEAR OSCILLATORY SECTIONS†

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A mechanical system with strongly non-linear oscillatory sections is considered. It is assumed that the oscillatory section has only one degree of freedom, and that the equations describing its motion are nearly Hamiltonian. A method is developed for deriving perturbed equations of motion for the oscillatory section, enabling mutual interaction to occur between non-linear oscillatory sections and enabling the motion of the system as a whole to be investigated by averaging methods. The use of the method to investigate the dynamics of the linkage of two material points in a motion with strongly non-linear longitudinal oscillations is considered.

THE AVERAGING method effectively enables one to investigate fundamental mechanisms in the dynamics of systems with weak non-linear interactions. Nevertheless, its application assumes a special form for the notation of the equations of perturbed motion in which the variables describing the motion are divided into rapidly and slowly varying ones. The problem of deriving perturbed equations of motion, which includes the problem of choosing variables to describe the motion, is strictly a problem for theoretical mechanics, and its solution to a large degree governs the success or failure of the investigation.

The introduction of "action-angle" variables is formally a complete solution to the problem of deriving perturbed equations of motion for systems whose equations of motion are near to integrable Hamiltonian systems. However, in practice these variables have not been widely used, because they are connected with the solution of a set of rather complex problems, and the representation of generalized coordinates and velocities in terms of "action-angle" variables is, in fact, associated with the use of infinite Fourier series.

For systems that are nearly conservative with a single degree of freedom, a method for constructing the first and higher approximations was presented in [1] which did not require the explicit solution of the unperturbed equations. In a number of problems the absence in this method of an explicit form of the perturbed equations of motion leads to excessively complicated transformations and calculations, and also impedes the choice of motion variables that are convenient for specific systems. The difficulties associated with the integration of implicitly specified functions are overcome in this method by averaging over the solution that is being generated. However, in many cases this operation cannot be performed in analytic form and therefore has to be performed at each step of the integration. The method presented below enables one in many cases to simplify the processes of constructing the equations for the first approximation for designated systems, and also significantly widens the range of systems investigated.

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1. STATEMENT OF THE PROBLEM

The motion of an oscillatory section is described by the equations

$$\ddot{g} = -T(\mathbf{x}, g) + \varepsilon F(\mathbf{x}, \mathbf{y}, g, \dot{g}) \tag{1.1}$$

where g is the generalized coordinate of the oscillating link, $\mathbf{x}$ and $\mathbf{y}$ are, respectively, the slow and fast variable vectors describing the motion of other parts of the system

$$\begin{aligned} \dot{\mathbf{x}} &= \varepsilon \mathbf{X}(\mathbf{x}, \mathbf{y}, g, \dot{g}) \\ \dot{\mathbf{y}} &= \omega(\mathbf{x}, \mathbf{y}) + \varepsilon \mathbf{Y}(\mathbf{x}, \mathbf{y}, g, \dot{g}) \end{aligned} \tag{1.2}$$

and ε is the small parameter.

In unperturbed motion ($\varepsilon = 0$) the motion of the oscillatory section has constant parameters and does not depend on the motion of the system. In perturbed motion ($\varepsilon \neq 0$) there is a weak non-linear coupling between the motions of the oscillating section and other parts of the system.

The motion of the oscillating section is close to the motion of a conservative system with one degree of freedom. In unperturbed motion the solution of Eqs (1.1) therefore reduces to quadratures. In particular cases, when the quadratures are explicitly solvable and one can construct a general solution for the oscillatory section which does not contain implicitly specified functions, the scheme for deriving the equations of perturbed motion is fairly simple [2]. This paper considers cases when it is not possible to construct a general solution for the motion of the oscillatory section which does not contain implicitly specified functions.

2. FUNDAMENTAL PROPERTIES OF THE UNPERTURBED MOTION OF AN OSCILLATORY SECTION

In unperturbed motion $\varepsilon = 0$, $\mathbf{x} = \text{const}$, and Eqs (1.1) have the first integral

$$h = \frac{1}{2}\dot{g}^2 + \Pi(\mathbf{x}, g), \quad \Pi(\mathbf{x}, g) = \int T(\mathbf{x}, g)\,dg \tag{2.1}$$

where h is a constant. The solution is given by the quadrature

$$t - t_0 = \int \frac{dg}{\sqrt{f(\mathbf{x}, g)}}, \quad f(\mathbf{x}, g) = 2h - 2\Pi(\mathbf{x}, g) \tag{2.2}$$

In most cases it is impossible to represent g as an explicit function of t. We know [3] that for systems with one degree of freedom in a force field two types of motion are possible: librational and limiting. For physical representations of the behaviour of an elastic system we will only consider librational motion, i.e. motion in which g varies in a periodic manner between boundary values g_1 and g_2 ($g_1 < g_2$), which are simple roots of the equation $f(\mathbf{x}, g) = 0$. Then the variation of g can be represented in the form

$$g = a - b\Phi(w(t)), \quad a = (g_1 + g_2)/2, \quad b = (g_2 - g_1)/2$$

where a and b are the mean and amplitude of the oscillation, respectively, $\Phi(w)$ is a function periodic in w varying over the range $[-1, 1]$, and w is the phase of the oscillation and increases monotonically with time. The functions Φ and w are related by the equation

$$b^2(d\Phi/dw)^2(dw/dt)^2 = 2h - 2\Pi \tag{2.3}$$

Two methods are usually used [3, 4] to determine the functions Φ and w. The first method sets $\Phi_1(w_1) = \cos(w_1)$, so that

$$\dot{w}_1 = \theta_1(g) = \left(\frac{f(\mathbf{x},g)}{(g-g_1)(g_2-g)}\right)^{1/2} = \left(\frac{2h-2\Pi}{b^2\sin^2 w_1}\right)^{1/2} \tag{2.4}$$

$$g = a - b\cos w_1$$

The second method sets $w_2 = (2\pi/\omega_{11})(t-t_0)$, where

$$\omega_{11} = 2\int_{g_1}^{g_2} \frac{dg}{\sqrt{f(x,g)}} = 2\int_0^{\pi} \frac{dw_1}{Q_1(g)} \tag{2.5}$$

is the oscillation period. Then

$$\frac{d\Phi_2}{dw_2} = \pm\frac{\omega_{11}}{2\pi b}\sqrt{f(\mathbf{x},g)}, \quad g = a - b\Phi_2(w_2) \tag{2.6}$$

where the plus sign corresponds to decreasing g and the minus sign to increasing g, and the function $\Phi_2(w_2)$ can be represented in the form of a Fourier series

$$\Phi_2(w_2) = \sum_{n=0}^{\infty} B_n \cos nw_2, \quad \sum_{n=0}^{\infty} B_n = 1$$

Since the amplitude of the oscillations b, the mean value a, and the energy constant h are related by the relation

$$h = \Pi(\mathbf{x}, a+b) = \Pi(\mathbf{x}, a-b) \tag{2.7}$$

the general form of the oscillation of the section can be described by the formulae

$$g = g(\mathbf{x}, h, w), \quad g(\mathbf{x}, h, w+\pi_0) = g(\mathbf{x}, h, w) \tag{2.8}$$
$$\dot{g} = (\partial g/\partial w)\dot{w}, \quad \dot{w} = Q(\mathbf{x},h,g) = [f(\mathbf{x},g)(\partial g/\partial w)^{-2}]^{1/2}$$

where π_0 is the period of the oscillations of g with respect to w.

3. THE EQUATIONS OF PERTURBED MOTION

The derivation of the perturbed equations of motion is based on an external principle of the method of varying arbitrary constants: the specification of a form of the equations that is convenient for the investigation, and variation of the parameters of this form. Because the oscillations of the section are described by an amplitude and phase and in the general case are described by formulae (2.8), we take h and w to be the new variables and take relations (2.8) to be the change of variable formulae. Differentiating (2.1) with respect to time, we obtain from (1.1)

$$\dot{h} = Q(\partial g/\partial w)\varepsilon F + (\partial\Pi/\partial\mathbf{x})\dot{\mathbf{x}} \tag{3.1}$$

The second equality of (2.8) gives

$$\dot{w} = Q - (\dot{h}\partial g / \partial h + \dot{\mathbf{x}}\partial g / \partial \mathbf{x})(\partial g / \partial w)^{-1} \tag{3.2}$$

Consequently, in the general case the system of perturbed equations of motion for the oscillatory rod is given by Eqs (3.1) and (3.2).

From the derivation of the equations it is clear that instead of h one could have chosen any other constant of the unperturbed motion which characterizes the amplitude of the oscillations of the section. In particular, it can be some function of g_1 and g_2. The coupling between g_1, g_2, h and $\mathbf{x}$ is given by relation (2.7). Consequently

$$\dot{g}_i = \left[\varepsilon F \theta \frac{\partial g}{\partial w} + \dot{\mathbf{x}}\left(\frac{\partial \Pi(\mathbf{x},g)}{\partial \mathbf{x}} - \frac{\partial \Pi(\mathbf{x},g_i)}{\partial \mathbf{x}}\right)\right]\left(\frac{\partial \Pi(\mathbf{x},g_i)}{\partial g_i}\right)^{-1} \tag{3.3}$$

In the representation $g = a - b\cos w_1$, where for unperturbed motion w_1 is given by Eq. (2.4), the equations for the oscillations of the section can be written in the form

$$\dot{b} = \left\{\varepsilon Q_1 bF \sin w_1 + \dot{\mathbf{x}}\left[\frac{\partial \Pi(\mathbf{x},g)}{\partial \mathbf{x}} - \frac{\partial \Pi(\mathbf{x},g_2)}{\partial \mathbf{x}} - \frac{\partial a}{\partial \mathbf{x}}\frac{\partial \Pi(\mathbf{x},g_2)}{\partial g_2}\right]\right\}\left\{\frac{\partial \Pi(\mathbf{x},g_2)}{\partial g_2}\left(1 + \frac{\partial a}{\partial b}\right)\right\}^{-1} \tag{3.4}$$

$$\dot{w}_1 = Q_1 + \left(\dot{b}\cos w_1 - \frac{\partial a}{\partial \mathbf{x}}\dot{\mathbf{x}} - \frac{\partial a}{\partial b}\dot{b}\right)(b \sin w_1)^{-1}$$

This form of the equations for the oscillatory section is useful because the oscillations of the section are described by trigonometric functions. In addition, in the general case, the dependence of Q_1 on w_1 in fact restricts the applications of averaging operators to the case of a single fast variable w_1.

In the representation $r = a - b\Phi_2(w_2)$, where $w_2 = (2\pi/\omega_{11})(t - t_0)$ and ω_{11} is the oscillatory period of the section in unperturbed motion, the equations of the oscillatory section differ from (3.4) in that the first term in the braces is replaced by $2\pi\omega_{11}^{-1}\varepsilon bF\partial\Phi_2/\partial w_2$, while the second equation is replaced by

$$\dot{w}_2 = \frac{2\pi}{\omega_{11}}(t - t_0) - \left(\dot{b}\Phi_2(w_2) - \frac{\partial a}{\partial \mathbf{x}}\dot{\mathbf{x}} - \frac{\partial a}{\partial b}\dot{b}\right)\left(b\frac{\partial \Phi_2(w_2)}{\partial w_2}\right)^{-1}$$

where Φ_2 and ω_{11} are unchanged in the perturbed motion, and Φ_2, w_{11} and $\partial\Phi_2/\partial w_2$ are determined from (2.5) and (2.6) for initial values of the parameters $\mathbf{x}, h$.

This form of the equations is useful in that the phase of the oscillations in the zeroth approximation is a linear function of time, which enables one to apply the most fully developed and simplest averaging algorithms for autonomous rotating systems [5], i.e. to apply averaging over the angular variables. The fixed functions $\Phi_2(w_2)$ and $\partial\Phi_2/\partial w_2$ are periodic and can be numerically calculated to practically any desired degree of accuracy.

The method presented has been used to perform a short derivation of the perturbed equations of Keplerian motion [6].

4. EXAMPLE

Consider the motion of two material points in a Newtonian field of force, the points being connected by a light thread whose elastic properties are described by Hooke's law. It is assumed that the trajectory of the centre of mass is an unperturbed Keplerian orbit. The equations of motion of the system (the link) about the centre of mass can be written in the form [7]

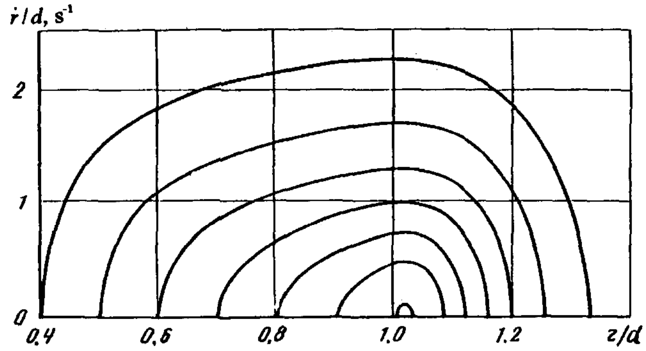

FIG. 1.

$$\dot{\psi}=\frac{rF_3\sin\varphi}{L\sin\theta},\quad \dot{\theta}=\frac{rF_3\cos\varphi}{L},\quad \dot{L}=rF_2,\quad \dot{\varphi}=\frac{L}{r^2}-\dot{\psi}\cos\theta$$

$$\ddot{r}-\frac{L^2}{r^3}+\delta c_m(r-d)=F_1,\quad \delta=\begin{cases}0, & r<d\\ 1, & r\geqslant d\end{cases}$$

$$F_1=\frac{\partial U}{\partial r},\quad F_2=\frac{1}{r}\frac{\partial U}{\partial\varphi},\quad F_3=\frac{1}{r\sin\varphi}\frac{\partial U}{\partial\theta} \tag{4.1}$$

$$U=\frac{1}{2}\frac{\mu}{R^2}r^2(3\cos^2\gamma-1),\quad c_m=c\left(\frac{1}{m_1}+\frac{1}{m_2}\right)$$

$$L=|\mathbf{r}\times\dot{\mathbf{r}}|,\quad \mathbf{r}=\mathbf{R}_2-\mathbf{R}_1$$

Here φ, ψ and θ are Euler angles describing the orientation of the link in a non-rotating "perigee" system of coordinates, $\mathbf{R}_2$ and $\mathbf{R}_1$ are the radius-vectors of the material points with respect to the attracting Newtonian centre, c is the stiffness of the thread, m_1 and m_2 are the masses of the material points, F_1, F_2 and F_3 are projections of the Newtonian accelerating field of force along the axes of the moving coordinate system, R is the distance from the centre of mass of the link to the attracting centre, μ is the gravitational constant, and γ is the angle between the vector $\mathbf{r}$ and the local vertical.

The method under consideration was used in [7] to investigate rapid link rotations in the cases of small-amplitude longitudinal oscillations and in motions which do not deviate from the link, i.e. in the case of quasi-linear oscillations. Here we consider rapid rotations of the link in motion regimes with large-amplitude longitudinal oscillations with possible deviation from the link. The phase portrait of unperturbed longitudinal oscillations is shown in Fig. 1 for $c_m=50\,\text{s}^{-2}$, $L/d=1\,\text{s}^{-1}$ (the lower part of the phase portrait being the mirror-image of the upper). It is clear that the oscillations are linear in nature only for small amplitudes, while for large amplitudes the character of the oscillations is significantly non-linear, i.e. the period and shape of the oscillations are amplitude-dependent.

We take as the independent variable the angular quantity φ^0, whose variation with time corresponds to the variation of the angle of pure rotation $\dot{\varphi}^0=L/r^2$. Then the longitudinal oscillations in the unperturbed motion are described by the Binet equations [3]

$$L^2u^2(d^2u/d\varphi^{02}+u)=\delta c_m(1/u-d)\quad (u=1/r)$$

while the energy integral becomes

$$h=\frac{L^2}{2}\left(\left(\frac{du}{d\varphi^0}\right)^2+u^2\right)+\Pi_1\left(\frac{1}{u}\right),\quad \Pi_1=\int\delta c_m(r-d)dr$$

Using the representation

$$u = a_u - b_u\Phi_3(w_3), \quad w_3 = 2\pi/\omega_{12}\cdot(\varphi^0 - \varphi_0^0)$$

$$\omega_{12} = 2L\int_{r_1}^{r_2}\frac{dr}{r^2\sqrt{f(L,r)}}$$

(where ω_{12} is the period of longitudinal oscillations with respect to "time" φ^0), we obtain the equations of the perturbed motion of the link in the form

$$\frac{d\psi}{d\varphi^0} = \frac{F_3\sin\varphi}{u^3L^2\sin\theta}, \quad \frac{d\theta}{d\varphi^0} = \frac{F_3\cos\varphi}{u^3L^2}$$

$$\frac{dL}{d\varphi^0} = \frac{F_2}{u^3L}, \quad \frac{d\alpha}{d\varphi^0} = -\frac{d\psi}{d\varphi^0}\cos\theta$$

$$\frac{db_u}{d\varphi^0} = \left\{\frac{1}{u^2}\frac{2\pi}{\omega_{12}}F_1b_u\frac{\partial\Phi_3}{\partial w_3} - \frac{dL}{d\varphi^0}\left[L(u_2^2 - u^2) + \right.\right.$$

$$\left.\left. + \frac{\partial a_u}{\partial L}\frac{\partial V(u_2)}{\partial u_2}\right]\right\}\left\{\frac{\partial V(u_2)}{\partial u_2}\left(1 + \frac{\partial a_u}{\partial b_u}\right)\right\}^{-1} \tag{4.2}$$

$$\frac{dw_3}{d\varphi^0} = \frac{2\pi}{\omega_{12}} - \left[\frac{\partial b_u}{\partial\varphi^0}\left(\Phi_3(w_3) - \frac{\partial a_u}{\partial b_u}\right) - \frac{\partial a_u}{\partial L}\frac{dL}{d\varphi^0}\right]\left(b_u\frac{\partial\Phi_3}{\partial w_3}\right)^{-1}$$

$$\left(\varphi = \varphi^0 + \alpha, \quad V(u) = \Pi_1 + \frac{1}{2}L^2u^2\right)$$

For non-resonant motion regimes (ω_{12} and 2π are rationally incommensurable) the equations for the first approximation obtained by averaging Eqs (4.2) over φ^0 and w_3 have the form

$$\frac{d\theta}{d\varphi^0} = N_1\sin\theta\cos(\nu-\psi)\sin(\nu-\psi), \quad \frac{d\psi}{d\varphi^0} = N_1\cos\theta\sin^2(\nu-\psi)$$

$$\frac{d\alpha}{d\varphi^0} = -N_1\cos^2\theta\sin^2(\nu-\psi), \quad \frac{dL}{d\varphi^0} = 0, \quad \frac{db_u}{d\varphi^0} = 0 \tag{4.3}$$

$$\left(N_1 = -\frac{3}{2}\frac{\mu}{p^3}\frac{r_*^4}{L^2}(1+e\cos\nu)^3, \quad r_*^4 = \frac{1}{2\pi}\int_0^{2\pi}\frac{dw_3}{(a_u - b_u\Phi_3(w_3))^4}\right)$$

Here ν is the true anomaly, e is the eccentricity, p is the focal parameter of the orbit of the centre of mass, and it is assumed that at the initial time $\varphi = \varphi^0$, i.e. $\alpha_0 = 0$. The variation of ν does not depend on the relative motion and is given by the equation

$$d\nu/dt = (\mu/p^3)^{1/2}(1+e\cos\nu)^2 \tag{4.4}$$

while the relation between t and φ^0 (ν and φ^0) is the same as for the unperturbed motion

$$d\varphi^0/dt = Lu^2 \tag{4.5}$$

Keeping the order of the approximation with respect to the small quantity

$$\varepsilon = \frac{\mu}{p^3}\frac{r_*^4}{L^2} \ll 1,$$

from Eqs (4.3)–(4.5) we obtain

$$\begin{aligned}
&\frac{d\theta}{d\nu}=N_0(1+e\cos\nu)\sin\theta\cos(\nu-\psi)\sin(\nu-\psi)\\
&\frac{d\psi}{d\nu}=N_0(1+e\cos\nu)\sin^2(\nu-\psi)\cos\theta\\
&\frac{d\alpha}{d\nu}=-N_0(1+e\cos\nu)\cos^2\theta\sin^2(\nu-\psi)\\
&N_0=-\frac{3}{2}\left(\frac{\mu}{p^3}\right)^{1/2}\frac{r_*^4}{r^{*2}}\frac{1}{L},\quad r^{*2}=\frac{1}{2\pi}\int_0^{2\pi}\frac{dw_3}{(a_u-b_u\Phi_3(w_3))^2}
\end{aligned}\tag{4.6}$$

It follows from Eqs (4.6) that the evolution of the direction of the angular momentum of the link can, to a first approximation, be calculated in the same way as the evolution of the angular momentum of a dumb-bell with a rod of length $(r_*^4/r^{*2})^{1/2}$.

We determine the equations describing the main evolutionary effects of the motion of the link by averaging (4.6) over ν

$$\frac{d\theta}{d\nu}=0,\quad \frac{d\psi}{d\nu}=\frac{1}{2}N_0\cos\theta,\quad \alpha=(\psi-\psi_0)\cos\theta \tag{4.7}$$

From Eqs (4.6) and (4.7) one can conclude that the amplitude of the longitudinal oscillations, both in the case of motion without deviation from the link, and with deviation from the link, do not qualitatively change the nature of the evolution of the parameters of the link motion, and only govern the rate of precession of the angular momentum in the secular motion and the amplitude of the deviation of its motion from uniform precession.

We remark that the use of form (3.4) of the equations of perturbed motion and averaging over the variables w_1 and φ in the case of strongly non-linear oscillations leads to false results.

REFERENCES

1. VOLOSOV V. M. and MORGUNOV B. I., *The Averaging Method in the Theory of Non-linear Oscillatory Systems.* Izd. MGU, Moscow, 1971.
2. MOISEYEV N. N., *Asymptotic Methods of Non-linear Mechanics.* Nauka, Moscow, 1981.
3. PARS L., *Analytical Dynamics.* Nauka, Moscow, 1971.
4. LUR'YE A. I., *Analytical Mechanics.* Fizmatgiz, Moscow, 1961.
5. GREBENIKOV Ye. A., *The Averaging Method in Applied Problems.* Nauka, Moscow, 1986.
6. PIROZHENKO A. V., Equations of constrained motion for a material point on an elastic constraint. *Prikl. Mekh.* **26**, 126–129, 1990.
7. ALPATOV A. P., BELONOZHKO P. A., PIROZHENKO A. V. and SHABOKHIN V. A., On the evolution of the rotational motion of a link between two bodies in orbit. *Kosmich. Issled.* **28**, 692–701, 1990.

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